Optimal sampling design for approximation of stochastic Itô integrals with application to the nonlinear Lebesgue integration *

Extended abstract
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Paweł Przybyłowicz

AGH University of Science and Technology, Department of Applied Mathematics,
Al. Mickiewicza 30, 30-059 Krakow, Poland,
E-mail: pprzybyl@agh.edu.pl

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In the paper we deal with the problem of optimal approximation of stochastic Itô integrals of the following form

$$I(f, B) = \int_0^T f(t, B_t) dB_t,$$

where $T > 0$ is a given number, a function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies certain regularity conditions and $(B_t)_{t \in [0, T]}$ is a one dimensional Brownian motion on some probability space $(\Omega, \Sigma, \mathbb{P})$.

Since in most cases explicit value of the integral (1) is not available, we must consider approximation schemes. We are interested in algorithms which use only discrete values of the driving Brownian motion and which have minimal error. Hence, there arises a problem of efficient sampling of the process $B$ in order to obtain estimator of (1) with the (asymptotic) error as small as possible.

Recently the problem of optimal approximation of stochastic integrals has been widely studied in the literature. In particular, the problem (1) was investigated in [3], [11], [12] and [17]. In these papers the error of an algorithm was measured in the average case sense with respect to the Brownian motion and in the worst case sense with respect to the function class of mappings $f$.

In this work for the problem (1) we use different approach to that used in mentioned articles. Namely, instead of fixing cardinality of the information used and varying $f$ over a whole class of functions we fix function $f$ and investigate how fast the error of an algorithm can converge to zero with cardinality tends to infinity. Hence, we investigate approximation of (1) in the so called asymptotic setting with respect to integrands $f$ (see [16] for further discussion of differences between the worst case and asymptotic models).

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Since the value of the integral (1) for integrands \( f = f(t, y) \) is a solution of a system of stochastic differential equations (SDEs), the problem under consideration is connected with the problem of optimal pointwise approximation of solutions of such systems. Hence, we can use the well-known methods, such as Euler or Milstein algorithms, in order to obtain an approximation of the value of (1), see [6]. Moreover, in the article [9] author considered problem of optimal pointwise approximation of scalar stochastic differential equations. However, since (1) is the value of the solution of the scalar SDE only in the case when \( f = f(t) \), results from [9] cannot be directly used for our problem (1) with \( f = f(t, y) \). Furthermore, in the paper we impose weaker assumptions on a function \( f \) than in [6] and [9], and we give exact rate of convergence of the minimal errors together with asymptotic constants.

Depending on the sampling method for trajectories of the Brownian motion \( B \), we consider two classes of algorithms \( \chi^\text{eq} \) and \( \chi^\text{noneq} \). Class \( \chi^\text{eq} \) contains methods which are based on an equidistant discretization of the interval \([0, T]\). Algorithms which use observations of trajectories of the process \( B \) at points chosen in adaptive way with respect to \( f \) (which are not necessary equidistant), belong to a wider class \( \chi^\text{noneq} \).

Main result of the paper, states that for fixed \( f \) and in the class \( \chi^\text{noneq} \) it holds

\[
\lim_{n \to +\infty} n \cdot \inf_{A = \{A_k\}_{k \geq 1} \in \chi^\text{noneq}} ||I(f, B) - A_n(f, B)||_2 = \frac{1}{\sqrt{12}} \left( \int_0^T \left( \mathbb{E}(\mathcal{Y}_t(f))^2 \right)^{1/3} dt \right)^{3/2}, \tag{2}
\]

where \( \mathcal{Y}_t(f) = f^{(1,0)}(t, B_t) + \frac{1}{2} f^{(0,2)}(t, B_t) \), \( f^{(1,0)} \) and \( f^{(0,2)} \) are partial derivatives of \( f \) and \( || \cdot ||_2 \) is the \( L^2(\Omega) \) norm. By taking infimum we mean that we are choosing the mappings \( \{A_k\}_{k \geq 1} \) along with the discretization points in the best possible way. For a smaller class \( \chi^\text{eq} \) we have

\[
\lim_{n \to +\infty} n \cdot \inf_{A = \{A_k\}_{k \geq 1} \in \chi^\text{eq}} ||I(f, B) - A_n(f, B)||_2 = \frac{T}{\sqrt{12}} \left( \int_0^T \mathbb{E}(\mathcal{Y}_t(f))^2 dt \right)^{1/2}. \tag{3}
\]

The order of convergence for both classes is \( n^{-1} \), but the asymptotic constant in (3) may be considerably larger than the asymptotic constant in (2). The asymptotically optimal schemes are defined by the conditional expectation of the Wagner–Platen algorithm under given values of Brownian motion at discretization points. For the Milstein scheme we show that it behaves suboptimally in classes of algorithms \( \chi^\text{eq} \) and \( \chi^\text{noneq} \).

It is known from [3] and [17] that the problem of approximation with minimal possible error of the stochastic integrals (1) is closely related to the problem of optimal approximation of Lebesgue integrals of the following form

\[
\mathcal{J}(f, B) = \int_0^T f(B_t) dt, \tag{4}
\]

with \( f : \mathbb{R} \to \mathbb{R} \). Applying results obtained for (1) to the problem (4) we give the exact rate of convergence of the minimal errors that can be achieved by arbitrary algorithm from classes \( \chi^\text{eq} \) and \( \chi^\text{noneq} \). We show that in class \( \chi^\text{noneq} \) the following holds

\[
\lim_{n \to +\infty} n \cdot \inf_{A = \{A_k\}_{k \geq 1} \in \chi^\text{noneq}} ||\mathcal{J}(f, B) - A_n(f, B)||_2 = \frac{1}{\sqrt{12}} \left( \int_0^T \mathbb{E}(f'(B_t))^2 dt \right)^{3/2}, \tag{5}
\]
while in class $\chi^{eq}$ we have that

$$\lim_{n \to +\infty} n \cdot \inf_{\mathcal{A} = \{A_k\}_{k \geq 1} \in \chi^{eq}} \| J(f, B) - A_n(f, B) \|_2 = \frac{T}{\sqrt{12}} \left( \int_0^T E(f'(B_t))^2 dt \right)^{1/2}. \quad (6)$$

Hence, the optimal rate of convergence $n^{-1}$ is the same as for the stochastic Itô integration (1). We also define algorithms which are asymptotically optimal in the respective classes.

References


