A few results concerning the Hurwitz stability of polytopes of complex polynomials

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ABSTRACT

Let \(C_{f_1,...,f_m}\) be a polytope generated by complex polynomials \(f_1,...,f_m\) whose degrees differ at most by one. The main goal of this note is to provide a tool for verifying whether a polynomial family \(C_{f_1,...,f_m}\) is stable. The note extend a few important results of the robust stability theory (the Edge Theorem given by Bartlett et al. [5], its generalizations proposed by Sideris and Barmish [7] and Fu and Barmish [6] and the eigenvalue criterions of Białas [4,11]) to more general cases concerning complex polynomial families without degree-invariant assumptions. Numerical examples are presented to complete and illustrate the results.

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1. Introduction

In this note, we reconsider an over 30 years old problem of determining the localization of zeros of polynomial families. We focus our attention on the Hurwitz stability of polytopes of polynomials. This issue has been widely investigated in the past 30 years and the literature on this subject is very abundant. We mention here only a few results which are closely related to this paper; for the rest of them see monographs [1–3] and the references therein.

One of the first works concerning the Hurwitz stability problem for polytopes of polynomials is Białas [4] where segments of real polynomials were considered. This paper provide the so-called...
eigenvalue criterion stating that a segment of real polynomials is Hurwitz stable if and only if a certain matrix formed from the coefficients of two vertex polynomials has no negative eigenvalues. Three years later, Bartlett et al. [5] proved the well-known Edge Theorem. These two results combined together make possible to verify the Hurwitz stability of polytopes consisting of real polynomials of the same degrees.

We nowadays know at least a few very important generalizations of the Edge Theorem: Fu and Barmish [6], for example, provide a version of the Edge Theorem that holds for complex polynomials of the same degrees and for domains whose complements are pathwise connected on the Riemann sphere (simple connectedness of a domain implies pathwise connectedness of its complement on the Riemann sphere; see Fu and Barmish [6]). In turn, Sideris and Barmish proved in [7] that for polytopes of real polynomials of different degrees and for domains satisfying the condition that each component of their complement is pathwise connected and contains at least a point on the real axis, the Edge Theorem also stays true.

In this paper we deal with a polytope of complex polynomials of different degrees. We will consider a case in which vertex polynomials generating the polytope differ in degrees at most by one. One of the main results of this work is to show that in such a case, a polytope is Hurwitz stable if and only if all its edge polynomials are. At first glance, this result may seem to be much weaker than these mentioned above – it can only be applied for a domain being the open left-half of the complex plane. However, we show that the general variant of the edge theorem holding for polytopes of real polynomials of different degrees and for much more general domains than the open left-half of the plane, does not hold in the complex case.

The paper is organized as follows. After preliminary Section 2 we study in Section 3 the Hurwitz stability of polytopes of complex polynomials. Our main goal in that section is to show that the Hurwitz stability of a polytope generated by \( m \) \((m \geq 3)\) complex polynomials whose degrees differ by one is equivalent to the Hurwitz stability of all its edge polynomials. Next, in Section 4, we give a necessary and sufficient condition for the Hurwitz stability of a segment of complex polynomials (of arbitrary degrees). In Section 5, we present two numerical examples which illustrate our main results. Finally, in Section 6, we summarize our main results and provide some suggestions for future research.

2. Definitions and preliminary materials

2.1. Basic notation

Throughout this paper, \( \mathbb{R} \) and \( \mathbb{C} \) stand for the field of real and complex numbers, respectively. By \( \mathbb{R}^{n \times n} \) (resp. \( \mathbb{C}^{n \times n} \)) we will denote the space of square matrices of dimension \( n \) with real (resp. complex) entries. \( \Re(s) \), \( \Im(s) \), \( |s| \) and \( \bar{s} \) stand for the real part, the imaginary part, the moduli and the complex conjugate of a complex number \( s \); \( i \) stands for the imaginary unit. The degree of a polynomial will be denoted by \( \deg(\cdot) \).

2.2. Sets of polynomials

\( \mathcal{P}_n(\mathbb{R}) \) (resp. \( \mathcal{P}_n(\mathbb{C}) \)) shall denote the linear space of real (resp. complex) polynomials of degree less than or equal to \( n \). It will be convenient to identify polynomials with vectors of their coefficients, i.e.:

\[
\mathcal{P}_n(F) \ni (s \rightarrow a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) \sim \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix} \in F^{n+1},
\]

where \( F = \mathbb{R} \) or \( F = \mathbb{C} \); polynomials of degree less than \( n \) shall be identified with vectors of their coefficients completed with zeros to vectors of appropriate length. In addition, every vector of the space \( \mathbb{C}^{n+1} \) (and hence every polynomial) can be viewed as an element of \( (2n + 2) \)-dimensional real space \( \mathbb{R}^{2n+2} \).

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For a set (of polynomials) $A$, $\text{aff}(A)$ stands for the affine hull of $A$; $\overline{A}$, $\partial A$ and $A^c$ stand, respectively, for its relative closure, boundary and complement in $\text{aff}(A)$. A dimension of a set will be denoted by $\dim(\cdot)$ and should be understood as a dimension of the affine hull of the set.

2.3. Stable polynomials

A complex polynomial of degree $n$ ($n \geq 1$)

$$f(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$  \hspace{1cm} (1)

is said to be Hurwitz stable if $\Re (s_i) < 0$ ($i = 1, \ldots, n$). In the sequel, we will write stable instead of Hurwitz stable. A polynomial family $P$ is said to be stable if $P \subset \mathcal{H}$, where $\mathcal{H}$ denotes the entire family of stable polynomials. $\overline{f}$ will denote the complex conjugate of polynomial (1), i.e.:

$$\overline{f}(s) = a_n \overline{s}^n + a_{n-1} \overline{s}^{n-1} + \cdots + \overline{a}_1 s + \overline{a}_0.$$

It is easy to see that if $f$ is a (real or complex) polynomial of degree $n$, then $ff$ is a real polynomial of degree $2n$ and $f \in \mathcal{H}$ if and only if $\overline{f} \in \mathcal{H}$.

For the set of polynomials $P$, $\mathcal{R}(P)$ shall denote its root space:

$$\mathcal{R}(P) = \{ s \in \mathbb{C} : \exists f \in P : f(s) = 0 \}.$$  

It is clear that the polynomial family $P$ is stable iff $\mathcal{R}(P) \subset \{ s \in \mathbb{C} : \Re (s) < 0 \}$.

2.4. The Routh–Hurwitz criterion

Let $H(f) \in \mathbb{C}^{n \times n}$ denote the Hurwitz matrix associated with polynomial (1), that is

$$H(f) = \begin{pmatrix}
    a_{n-1} & a_n & 0 & 0 & 0 & \cdots & 0 \\
    a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & \cdots & 0 & a_0
\end{pmatrix},$$  \hspace{1cm} (2)

where $a_j = 0$, for $j < 0$. The determinant of the Hurwitz matrix can be calculated using the Orlando’s formula:

$$\det H(f) = (-1)^{(n-1)/2} a_0 a_n^{n-1} \prod_{1 \leq k < l \leq n} (s_k + s_l),$$  \hspace{1cm} (3)

where $s_1, \ldots, s_n$ are the roots of the polynomial $f$; see Lancaster [8].

Let $\Delta_i[A]$ denote the $i$th leading principal minor of the square matrix $A = (a_{ij})_{i,j=1}^n$, that is

$$\Delta_i[A] = \det \begin{pmatrix}
    a_{i1} & \cdots & a_{ii} \\
    \vdots & \ddots & \vdots \\
    a_{i1} & \cdots & a_{ii}
\end{pmatrix}, \quad i = 1, \ldots, n.$$

The Routh–Hurwitz criterion states that a real polynomial $f$ having positive coefficients is stable iff $\Delta_i[H(f)] > 0$ for $i = 1, \ldots, n$.

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2.5. Polytopes, vertices, edges, faces

Consider now \( m \) (\( m \geq 2 \)) vectors \( v_1, \ldots, v_m \) of a linear space. The set

\[
C_{v_1,\ldots,v_m} = \left\{ \sum_{i=1}^{m} \alpha_i v_i : \alpha_i \geq 0 \ (i = 1, \ldots, m), \sum_{i=1}^{m} \alpha_i = 1 \right\}
\]

is a polytope generated by vectors \( v_1, \ldots, v_m \) called vertices, one-dimensional polytopes \( C_{v_1,\ldots,v_m} \cap H \), where \( H \) is a nontrivial supporting hyperplane of \( C_{v_1,\ldots,v_m} \), are called exposed edges, two-dimensional polytopes \( C_{v_1,\ldots,v_m} \cap H \) are called exposed faces.

We close this section with the following theorem (Sideris and Barmish [7]).

**Theorem 1.** Let \( \Omega \) be a polytope of complex polynomials of the same degree and let \( \mathcal{D} \subset \mathbb{C} \) be a region such that every component of \( \mathcal{D}^c \) is unbounded and pathwise connected. Then \( \Omega \) is \( \mathcal{D} \)-stable (i.e. has all its roots in \( \mathcal{D} \)) if and only if the exposed edges of \( \Omega \) are.

3. A stability condition for polytopes of complex polynomials whose degrees differ by one

Let us consider \( m \) (\( m \geq 3 \)) complex polynomials \( f_1, \ldots, f_m \in \mathcal{P}_n(\mathbb{C}) \) of the form

\[
\begin{align*}
f_1(s) &= a_{n,1}s^n + a_{n-1,1}s^{n-1} + \cdots + a_{1,1}s + a_{0,1}, \\
& \quad \vdots \\
f_m(s) &= a_{n,m}s^n + a_{n-1,m}s^{n-1} + \cdots + a_{1,m}s + a_{0,m}
\end{align*}
\]

and such that for some integer \( k \in \{1, \ldots, m-1\} \):

\[
n - 1 = \deg f_1 = \cdots = \deg f_k < \deg f_{k+1} = \cdots = \deg f_m = n.
\]

Assume also that the leading coefficients of polynomials (4) – equal respectively \( a_{n-1,1}, \ldots, a_{n-1,k}, a_{n,k+1}, \ldots, a_{n,m} \) – satisfy the following conditions:

(i) \( \forall \varepsilon \ (a_{n,i}a_{n-1,j}) > 0 \), for \( j = 1, \ldots, k; i = k+1, \ldots, m \); 
(ii) \( 0 \notin C_{a_{n-1,1},\ldots,a_{n-1,k}} \); 
(iii) \( 0 \notin C_{a_{n,k+1},\ldots,a_{n,m}} \).

Before we formulate the main result of this section, note that the above assumptions are not very restrictive. One can show that a segment linking two polynomials of different degrees and with the leading coefficients equal \( a_n \) and \( b_k \) is stable only if \( \Re \left( a_n\overline{b_k} \right) \geq 0 \) (see Białas and Góra [9]). In turn, conditions (ii) and (iii) are purely technical (but necessary) and guarantee that the polytopes of polynomials \( C_{f_1,\ldots,f_k} \) and \( C_{f_{k+1},\ldots,f_m} \) consist of polynomials of the same degrees equal \( n - 1 \) and \( n \), respectively (in other words, we know where, in the polytope, degree drop occurs).

**Theorem 2.** Under the notations and assumptions above, the polytope \( C_{f_1,\ldots,f_m} \) is stable if and only if all its exposed edges are.

Before we proof this theorem, we need a few auxiliary results.

**Lemma 3.** Let \( P \) be at least three dimensional set of complex polynomials that does not include a straight line. Then \( \mathcal{R}(P) \subset \mathcal{R}(\partial P) \).

**Proof.** Take any \( s_* \in \mathcal{R}(P) \). Then, there exists in \( P \) a polynomial, say \( f_* \), such that \( f_*(s_*) = 0 \). Let now \( \mathcal{P}_n^s(\mathbb{C}) \) be the linear subspace of \( \mathcal{P}_n(\mathbb{C}) \) consisting of polynomials vanishing at \( s_* \). It is clear that
dim $\mathcal{P}_n^k(\mathbb{C}) = 2n$. Hence, the set aff $P \cap \mathcal{P}_n^k(\mathbb{C})$ is a non-empty (it includes an element $f_s$), at least one-dimensional subspace of $\mathcal{P}_n(\mathbb{C})$ (because, by the assumptions, dim aff $(P) \geq 3$ and the dimension of the entire space $\mathcal{P}_n(\mathbb{C})$ equals $2n + 2$). It means by the assumption, that the set aff $(P) \cap \mathcal{P}_n^k(\mathbb{C})$ must leave the set $P$ piercing its (relative) boundary. In other words, $\mathcal{P}_n^k(\mathbb{C}) \cap \partial P \neq \emptyset$, which completes the proof. $\square$

From Lemma 3 one can easily draw the following

**Conclusion 1.** Let $P$ be at least three dimensional compact set of complex polynomials. Then $\mathcal{R}(P) = \mathcal{R}(\partial P)$.

**Lemma 4.** Let $f_1, \ldots, f_m \in \mathcal{P}_n(\mathbb{C})$ ($m \geq 3$) be given polynomials such that dim $C_{f_1, \ldots, f_m} \geq 2$. Then

\[
\mathcal{R}(C_{f_1, \ldots, f_m}) = \bigcup \mathcal{R}(C_{f_1, f_{i_1}, f_{i_2}}),
\]

and the sum is taken over all triples $1 \leq i_1 < i_2 < i_3 \leq m$ for which polytopes $C_{f_1, f_{i_1}, f_{i_2}}$ are contained in the exposed faces of the polytope $C_{f_1, \ldots, f_m}$.

**Proof.** If dim $C_{f_1, \ldots, f_m} = 2$ then the claim is obvious. Assume that dim $C_{f_1, \ldots, f_m} = k \geq 3$. Then the set $C_{f_1, \ldots, f_m}$ satisfies the assumptions of Conclusion 1 from which it follows that

\[
\mathcal{R}(C_{f_1, \ldots, f_m}) = \mathcal{R}(\partial C_{f_1, \ldots, f_m}).
\]

On the other hand, it can be easily shown that the relative boundary of a $k$-dimensional polytope is union of polytopes of dimension $k - 1$. Hence, using (6), we can replace the set $\partial C_{f_1, \ldots, f_m}$ with $(k - 1)$-dimensional polytopes contained in it and repeat the same reasoning for each of them. Finally, we obtain only two-dimensional polytopes contained in the exposed faces of the polytope $C_{f_1, \ldots, f_m}$. These polytopes, in turn, can be expressed as union of polytopes generated by triples of vertex polynomials $f_1, f_2, f_3$. This ends the proof. $\square$

Consider now three polynomials $f, g_1, g_2 \in \mathcal{P}_n(\mathbb{C})$ of the form

\[
f(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0,
\]

\[
a_i \in \mathbb{C} \ (i = 0, \ldots, n);
\]

\[
g_1(s) = b_{n,1} s^n + b_{n-1,1} s^{n-1} + \cdots + b_{1,1} s + b_{0,1},
\]

\[
b_{i,1} \in \mathbb{C} \ (i = 0, \ldots, n);
\]

\[
g_2(s) = b_{n,2} s^n + b_{n-1,2} s^{n-1} + \cdots + b_{1,2} s + b_{0,2},
\]

\[
b_{i,2} \in \mathbb{C} \ (i = 0, \ldots, n).
\]

**Lemma 5.** Let $f, g_1, g_2 \in \mathcal{P}_n(\mathbb{C})$ be polynomials (7)–(9) such that

\[
n - 1 = \deg f < \deg g_1 = \deg g_2 = n
\]

and whose leading coefficients – equal respectively $a_{n-1}, b_{n,1}, b_{n,2}$ – satisfy the following conditions

\[
\Re (b_{n,1} - a_{n-1}) > 0, \quad \Re (b_{n,2} - a_{n-1}) > 0
\]

and

\[
b_{n,1} \overline{b_{n,2}} \notin (-\infty, 0).
\]

Then the polytope $C_{f,g_1,g_2}$ is stable if and only if its (exposed) edges $C_{f,g_1}, C_{f,g_2}, C_{g_1,g_2}$ are.
The necessity is obvious. For the sufficiency, suppose that the sets \(C_f, \alpha, \beta\) are stable. Note, at the beginning, that the set \(C_f, \alpha, \beta\) can be written in the form
\[
\{\beta (\alpha g_1 + (1 - \alpha) g_2) + (1 - \beta) f : \alpha, \beta \in [0, 1]\}.
\]

Consider, for a fixed values of parameters \(\alpha, \beta \in [0, 1]\), the polynomial
\[
\psi_f (\alpha, \beta, s) = (\beta (\alpha g_1(s) + (1 - \alpha) g_2(s)) + (1 - \beta) f(s))
\times \left(\beta (\alpha g_1(s) + (1 - \alpha) g_2(s)) + (1 - \beta) \bar{f}(s)\right).
\]

It is clear that
\[
\psi_f (\alpha, 0, s) = f(s) \bar{f}(s) \tag{12}
\]
and that for every \(\alpha \in [0, 1]\) and \(\beta \in (0, 1]\) \(\psi_f (\alpha, \beta, \cdot)\) is a real polynomial of degree \(2n\). Writing this polynomial in its natural form we get
\[
\psi_f (\alpha, \beta, s) = A_{2n} s^{2n} + A_{2n-1} s^{2n-1} + \cdots + A_1 s + A_0,
\]
where the coefficients \(A_0, \ldots, A_{2n}\) are continuous real-valued functions of the coefficients of the vertex polynomials \(f, g_1, g_2\) and of the parameters \(\alpha, \beta\). In particular:
\[
A_{2n} = \beta^2 |\alpha b_{n,1} + (1 - \alpha) b_{n,2}|^2, \tag{13}
\]
\[
A_{2n-1} = 2\beta (1 - \beta) \Re ((\alpha b_{n,1} + (1 - \alpha) b_{n,2}) \overline{a_{n-1}}). \tag{14}
\]

Write now the Hurwitz matrix associated with the polynomial \(\psi_f (\alpha, \beta, \cdot)\):
\[
H (\psi_f (\alpha, \beta; \cdot)) =
\begin{pmatrix}
A_{2n-1} & A_{2n} & 0 & \cdots & 0 \\
A_{2n-3} & A_{2n-2} & A_{2n-1} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \cdots & A_0
\end{pmatrix} \tag{15}
\]

Hence, for \(k = 1, \ldots, 2n\):
\[
\Delta_k [H (\psi_f (\alpha, \beta; \cdot))] = \beta \Delta_k [H_f (\alpha, \beta)], \tag{16}
\]
where \(H_f (\alpha, \beta) \in \mathbb{R}^{2n \times 2n}\) is a matrix formed from matrix (15) by dividing its first row by \(\beta\). It now follows from (12)–(14) that the matrix \(H_f (\alpha, 0)\) written as a block matrix has the form
\[
H_f (\alpha, 0) = \begin{pmatrix}
2\Re (\alpha b_{n,1} + (1 - \alpha) b_{n,2}) \overline{a_{n-1}} & 0 \\
2\Re (a_{n-1} \overline{a_{n-2}}) |a_{n-1}|^2 & H (f \bar{f})
\end{pmatrix}, \tag{17}
\]
where \(O_{2 \times (2n-2)}\) is a \(2 \times (2n-2)\) zero matrix and \(X\) is some \((2n-2) \times 2\) matrix (it does not play any role in the sequel). From (17) we get that
\[
\Delta_1 [H_f (\alpha, 0)] = 2\Re (\alpha b_{n,1} + (1 - \alpha) b_{n,2}) \overline{a_{n-1}},
\]
\[
\Delta_2 [H_f (\alpha, 0)] = 2 |a_{n-1}|^2 \Re (\alpha b_{n,1} + (1 - \alpha) b_{n,2}) \overline{a_{n-1}}.
\]
and
\[ \triangle_{k+2} \left[ H_{\Psi_1, \Psi_2} (\alpha, 0) \right] = \triangle_2 \left[ H_{\Psi_1, \Psi_2} (\alpha, 0) \right] \triangle_k \left[ H (f) \right], \]

for \( k = 1, \ldots, 2n - 2 \). The stability of the polynomial \( f \) and assumption (10) imply that
\[ \triangle_k \left[ H_{\Psi_1, \Psi_2} (\alpha, 0) \right] > 0, \]

for \( k = 1, \ldots, 2n \) and for every \( \alpha \in [0, 1] \). It now follows from the continuity of the functions
\[ [0, 0] \times [0, \infty) \ni (\alpha, \beta) \mapsto \triangle_k \left[ H_{\Psi_1, \Psi_2} (\alpha, \beta) \right] \in \mathbb{R} \quad (k = 1, \ldots, 2n) \]

that there exist positive numbers \( \beta_1, \ldots, \beta_{2n} \) such that \( \triangle_k \left[ H_{\Psi_1, \Psi_2} (\alpha, \beta) \right] > 0 \), for every \( \beta \in (0, \beta_k) \), \( \alpha \in [0, 1] \) and \( k = 1, \ldots, 2n \). Putting \( \beta_* = \min \{ \beta_1, \ldots, \beta_{2n}, 1 \} \), from condition (16), we obtain the system of inequalities
\[ \triangle_k \left[ H (\Psi_{\Psi_1, \Psi_2} (\alpha, \beta_*; \cdot)) \right] > 0 \quad (k = 1, \ldots, 2n), \]

fulfilled for \( \alpha \in [0, 1] \) and \( \beta \in (0, \beta_*) \). By the Routh–Hurwitz criterion, it means that the polynomial \( \Psi_{\Psi_1, \Psi_2} (\alpha, \beta_*; \cdot) \) is stable for \( \alpha \in [0, 1] \) and \( \beta \in (0, \beta_*) \).

To complete the proof, take any \( \beta_0 \in (0, \beta_*) \) and consider the following sets:
\[ C_{f_{\Psi_1, \Psi_2}}^{(0)} = \{ \beta_0 g_1 + (1 - \alpha) g_2 : \alpha \in [0, 1], \beta \in [0, \beta_0] \}, \]
\[ C_{f_{\Psi_1, \Psi_2}}^{(1)} = \{ \beta (\alpha g_1 + (1 - \alpha) g_2) + (1 - \beta) f : \alpha \in [0, 1], \beta \in [\beta_0, 1] \}. \]

We now prove the stability of these sets.

The stability of family (18) follows from the preceding reasoning. To show the stability of set (19), note that it is a polytope generated by the following vertex polynomials: \( g_1, g_2, \beta_0 g_1 + (1 - \beta_0) f \) and \( \beta_0 g_1 + (1 - \beta_0) f \); all of the same degree \( n \). Moreover, it follows from the assumption and from the manner of choosing the value \( \beta_0 \), that the edge polynomials forming the relative boundary of set (19) are stable. By Theorem 1, it allows us to state that entire set (19) is also stable. Thus, since \( C_{f_{\Psi_1, \Psi_2}}^{(0)} \cup C_{f_{\Psi_1, \Psi_2}}^{(1)} = C_{f_{\Psi_1, \Psi_2}}, \) the proof is complete. \( \Box \)

**Lemma 6.** Let \( f, g_1, g_2 \in \mathcal{P}_n (\mathbb{C}) \) be polynomials (7)–(9) such that
\[ n - 1 = \deg g_1 = \deg g_2 < \deg f = n, \]
and whose leading coefficients equal respectively \( a_n, b_{n-1,1}, b_{n-1,2} \) satisfy the following conditions
\[ \Re \left( a_n \overline{b_{n-1,1}} \right) > 0, \quad \Re \left( a_n \overline{b_{n-1,2}} \right) > 0 \]

and
\[ b_{n-1,1} \overline{b_{n-1,2}} \neq (-\infty, 0). \]

Then the polytope \( C_{f, g_1, g_2} \) is stable if and only if its (exposed) edges \( C_{f, g_1}, C_{f, g_2}, C_{g_1, g_2} \) are.

**Proof.** The proof is analogous to that of Lemma 5. This time, we write the set \( C_{f, g_1, g_2} \) in the form
\[ C_{f, g_1, g_2} = \{ \beta f + (1 - \beta) (\alpha g_1 + (1 - \alpha) g_2) : \alpha, \beta \in [0, 1] \}. \]

Consider, for fixed values of parameters \( \alpha, \beta \in [0, 1] \), the polynomial
\[ \Phi_{f, g_1, g_2} (\alpha, \beta; s) = (\beta f (s) + (1 - \beta) (\alpha g_1 (s) + (1 - \alpha) g_2 (s))) \]
\[ \times \left( \beta \overline{f (s)} + (1 - \beta) (\alpha \overline{g_1 (s)} + (1 - \alpha) \overline{g_2 (s)}) \right). \]
As previously, \( \Phi_{f_1, g_2} (\alpha, \beta; \cdot) \) is a real polynomial whose degree equals \( 2n \), for \( \alpha \in [0, 1] \) and \( \beta \in (0, 1] \). Writing this polynomial in its natural form we get

\[
\Phi_{f_1, g_2} (\alpha, \beta; s) = A_{2n}s^{2n} + A_{2n-1}s^{2n-1} + \cdots + A_1s + A_0,
\]

where the coefficients \( A_0, \ldots, A_{2n} \) are continuous real-valued functions of the coefficients of the polynomials \( f, g_1, g_2 \) and of parameters \( \alpha, \beta \). In particular,

\[
A_{2n} = \beta^2 |a_n|^2, \\
A_{2n-1} = 2\beta (\beta - 1) \Re((\alpha b_{n-1,1} + (1 - \alpha) b_{n-1,2})\overline{a_n}) .
\]

Similarly as in the proof of Lemma 5, we show that

\[
\Delta_1 [H_{f_1, g_2} (\alpha, 0)] = 2\Re((\alpha b_{n-1,1} + (1 - \alpha) b_{n-1,2})\overline{a_n}) ,
\]

\[
\Delta_2 [H_{f_1, g_2} (\alpha, 0)] = 2 |\alpha b_{n-1,1} + (1 - \alpha) b_{n-1,2}|^2 \\
\times \Re((\alpha b_{n-1,1} + (1 - \alpha) b_{n-1,2})\overline{a_n})
\]

and

\[
\Delta_{k+2} [H_{f_1, g_2} (\alpha, 0)] = \Delta_2 [H_{f_1, g_2} (\alpha, 0)] \\
\times \Delta_k \left[H((\alpha g_1 + (1 - \alpha) g_2)(\alpha \overline{g_1} + (1 - \alpha) \overline{g_2}))\right] ,
\]

where \( H_{f_1, g_2} (\alpha, \beta) \in \mathbb{R}^{2n \times 2n} \) is a matrix formed from the Hurwitz matrix \( H(\Phi_{f_1, g_2} (\alpha, \beta; \cdot)) \) by dividing its first row by \( \beta \). Next, proceeding in the same manner as in case of Lemma 5, we complete the proof. \( \square \)

We can now justify Theorem 2.

**Proof of Theorem 2.** The necessity is obvious: all the exposed edges of a polytope are contained in it.

For the sufficiency, note that, by Lemma 4, for \( m \geq 3 \) the stability problem of the polytope \( C_{f_1, \ldots, f_m} \) can be replaced by the stability problem of the polytopes \( C_{f_{i_1}, f_{i_2}, f_{i_3}} \), for \( 1 \leq i_1 < i_2 < i_3 \leq m \). It allows us to restrict ourselves to the following three cases:

(i) \( \deg f_{i_1} = \deg f_{i_2} = \deg f_{i_3} \);  
(ii) \( n-1 = \deg f_{i_1} < \deg f_{i_2} = \deg f_{i_3} = n \);  
(iii) \( n-1 = \deg f_{i_1} = \deg f_{i_2} < \deg f_{i_3} = n \).

In case (i), the thesis follows from Theorem 1; in case (ii) it follows from Lemma 5, and in the last case from Lemma 6. This completes the proof. \( \square \)

### 4. A necessary and sufficient condition for the stability of a segment of complex polynomials

In this section, we focus our attention on the stability problem for segments of complex polynomials. This issue was taken up in many papers. The paper Białas [4] provide the so-called eigenvalue criterion saying that a segment generated by real polynomials, say \( f \) and \( g \), is stable iff the Hurwitz matrix \( H^{-1}(f)H(g) \) has no negative eigenvalues. Bose [12] and Hwang and Yang [13], in turn, considered segments of complex polynomials. The result that they obtained, based on the resultant matrix, was formulated only for polynomials of the same degree (see also Tits [14] for an additional comment on paper [13]).

Our aim now is to give some new necessary and sufficient condition for the stability of segments of complex polynomials. The condition does not require the assumption of the equality of degrees of vertex polynomials.

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Consider two complex polynomials
\begin{align}
f_1(s) &= b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0, \\
f_2(s) &= a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0, \quad a_n \neq 0
\end{align}
(22)
and assume that \( \deg f_1 = k \geq 1 \). Clearly, in case of \( n > k \), \( b_j = 0 \) for \( j = k + 1, \ldots, n \). Let for \( \alpha \in (0,1) \), \( P(\alpha) \) be a quadratic polynomial matrix of the form
\begin{align}
P(\alpha) &= \alpha^2 H \left( f_1^{(\alpha)} \right) + \alpha \left( 1 - \alpha \right) H \left( f_1^{(\alpha)} + f_1 \right) + (1 - \alpha)^2 H \left( f_2 \right).
\end{align}
(23)
The Hurwitz matrices occurring in (23) must have the same dimension equal 2n, that is, when forming matrices \( H \left( f_1^{(\alpha)} \right) \) and \( H \left( f_1^{(\alpha)} + f_1 \right) \) the missing coefficients of the polynomial \( f_1 \) (if any, that is, if \( \deg f_1 < \deg f_2 \)) are completed by zeros.

**Theorem 7.** Assume that complex polynomials (22) are stable and that in case \( \deg f_1 = \deg f_2 \) it holds
\begin{align}
a_n \tilde{b}_n &\notin (-\infty,0).
\end{align}
(24)
Then the polytope \( C_{f_1,f_2} \) is stable if and only if the matrix \( P(\alpha) \) is nonsingular for every \( \alpha \in (0,1) \).

**Proof.** Begin with the simple observation that \( f_1^{(\alpha)} + f_2, f_1, f_2 \in P_{2n}(\mathbb{R}) \). To prove the theorem, take any \( \tilde{f} \in C_{f_1,f_2} \); it is clear that \( \tilde{f} = \alpha f_1 + (1 - \alpha) f_2 \), for some \( \alpha \in [0,1] \). Hence and from (24) we obtain that for \( \alpha \in (0,1) \) the polynomial
\begin{align}
f(s) \tilde{f}(s) &= (\alpha f_1(s) + (1 - \alpha) f_2(s)) \left( \alpha f_1^{(s)} + (1 - \alpha) f_2^{(s)} \right)
\end{align}
(25)
is real and of degree 2n. By the assumption of the stability of \( f_2 \), if \( \alpha \) increases from 0 to 1, polynomial (25) stays stable as long as the Hurwitz matrix associated with this polynomial, which is in fact equal to \( P(\alpha) \), stays invertible (see e.g. Saydy et al. [15]). \( \square \)

**Remark 1.** One can easily deduce from the proof of Theorem 7 that if for some \( \alpha_0 \in (0,1) \) the matrix \( P(\alpha_0) \) is singular then the polynomial
\begin{align}
\alpha_0 f_1 + (1 - \alpha_0) f_2
\end{align}
(26)
has a root with non-negative real part. Moreover, if \( \alpha_0 \) is the smallest positive number for which matrix (23) is singular then polynomial (26) has a purely imaginary root.

In closing, we show how one can reformulate Theorem 7 to obtain an **eigenvalue criterion** for checking the stability of a segment of complex polynomials. Results of this kind can be found in papers of Ackermann and Barmish [16] and Białas [4,11] where the robust stability problem for segments of real polynomials were considered.

To do this, consider the block matrix \( W \in \mathbb{R}^{4n \times 4n} \) of the form
\begin{align}
W = \begin{pmatrix}
H \left( f_1^{(\alpha)} + f_1 \right) H^{-1} \left( f_2 \right) & H \left( f_1^{(\alpha)} \right) H^{-1} \left( f_2 \right) \\
-I_{2n} & 0_{2n}
\end{pmatrix},
\end{align}
(27)
where \( I_{2n} \) and \( 0_{2n} \) are the identity and zero matrices of dimension 2n, respectively. One can show that the characteristic polynomial of the matrix \( W \) is of the form
\begin{align}
det (\lambda I_{4n} - W) &= \begin{vmatrix}
\lambda I_{2n} - H \left( f_1^{(\alpha)} + f_1 \right) H^{-1} \left( f_2 \right) & I_{2n} \\
I_{2n} & \lambda I_{2n}
\end{vmatrix} \\
&= \det H^{-1} \left( f_2 \right) \det \left( H \left( f_1^{(\alpha)} \right) - \lambda H \left( f_1^{(\alpha)} + f_1 \right) + \lambda^2 H \left( f_2 \right) \right).
\end{align}
(28)
Putting now $\lambda = \frac{\alpha - 1}{\alpha}$ in (28), we get
\[
\det (\lambda I_n - W) = \det H^{-1} \left( f_2 \overline{f}_2 \right) \alpha^{-4n} \det P (\alpha).
\]
From the above, one can immediately draw the following

**Conclusion 2.** Under the assumptions of Theorem 7, the polytope $C_{f_1, f_2}$ is stable if and only if matrix (27) has no negative eigenvalue.

5. Examples

In this section, we give two simple examples illustrating the results developed in this paper.

**Example 1.** Given the segment $C_{f_1, f_2}$, where $f_1$ and $f_2$ are stable polynomials of the form:
\[
\begin{align*}
  f_1 (s) &= (1 + i) s + 1, \\
  f_2 (s) &= (1 - i) s^3 + (4 - 6i) s^2 + (5 - 15i) s + 2 - 14i.
\end{align*}
\]
To examine the stability of the set $C_{f_1, f_2}$ one can use Theorem 7 and check the nonsingularity of the matrix (23). Its determinant is a polynomial of degree 12 with respect to $\alpha$. When applying an algorithm based on the Sturm’s sequences to this polynomial one gets that it has a root in $(0, 1)$ and hence the set $C_{f_1, f_2}$ is not stable. Indeed, for $\alpha_0 = 0.92833$ the polynomial $\alpha_0 f_1 + (1 - \alpha_0) f_2$ has a purely imaginary root $s_0 = -2.6736i$.

**Example 2.** Given the polytope $C_{f_1, f_2, f_3}$ generated by polynomials:
\[
\begin{align*}
  f_1 (s) &= s^2 + (2 - 2i) s + 1 - 2i = (s + 1) (s + 1 - 2i), \\
  f_2 (s) &= s^2 + (2 + 4i) s + 1 + 4i = (s + 1) (s + 1 + 4i), \\
  f_3 (s) &= s + 1.
\end{align*}
\]
To determine whether the set $C_{f_1, f_2, f_3}$ is stable one can use Lemma 5 and check the stability of the edges $C_{f_1, f_2}, C_{f_1, f_3}$ and $C_{f_2, f_3}$. In this simple example (see a specific form of the polynomials $f_1, f_2$ and $f_3$), "direct calculations" lead to the following forms of the root spaces generated by the edge polynomials:
\[
\begin{align*}
  R (C_{f_1, f_2}) &= \{ s \in \mathbb{C} : s = -1 + \alpha i, \ \alpha \in [-4, 2] \}; \\
  R (C_{f_1, f_3}) &= \{ s \in \mathbb{C} : s = \alpha + 2i, \ \alpha \in (-\infty, -1] \cup \{-1\}; \\
  R (C_{f_2, f_3}) &= \{ s \in \mathbb{C} : s = \alpha - 4i, \ \alpha \in (-\infty, -1] \cup \{-1\}.
\end{align*}
\]
It means by Lemma 5 that the entire polytope $C_{f_1, f_2, f_3}$ is also stable. The form of the root space $R (C_{f_1, f_2, f_3})$ confirms the result:
\[
R (C_{f_1, f_2, f_3}) = \left\{ \frac{1+2\alpha-4\beta}{\alpha+\beta} : \alpha > 0, \beta > 0, \alpha + \beta < 1 \right\} \cup \bigcup_{1 \leq i < j \leq 3} R (C_{f_i, f_j}),
\]
(29)

In closing, recall that one of the most important extensions of the classical Edge Theorem was that given by Sideris and Barmish [7]. In their Theorem 2 they proved that for polytopes of real polynomials that can drop in degree by one at most, and for domains satisfying the condition that each connected component of their complement is unbounded and pathwise connected, the Edge Theorem also stays true. We now show that this result does not extend to the case of complex polynomials. To see this, return to Example 2 and consider the $\epsilon$-neighborhood of the root space of all the edge polynomials\footnote{An $\epsilon$-neighborhood of a subset $A$ of a metric space $X$ is a set $A_{\epsilon} = \{ x \in X : \text{dist} (x; A) < \epsilon \}$, where dist $(x; A)$ is a distance from the set $A$.}. 

denote it by $R_\epsilon$. Taking, for example, $\epsilon = 0.5$ we get the domain $R_{0.5}$ that satisfies the required condition: each connected component of its complement is unbounded and pathwise connected (see Fig. 1). In addition, when letting $\alpha = \beta = 0.125$ in (29), we obtain the root $s_0 = -4 - i$ such that $s_0 \in R(C_{f_1,f_2,f_3})$ and $s_0 \notin R_{0.5}$ (see Fig. 1). It means that the domain $R_{0.5}$ does not contain set (29) proving that Theorem 2 in Sideris and Barmish [7] does not extend to the case of polytopes of complex polynomials.

6. Concluding remarks

We have considered the Hurwitz stability problem for polytopes $C_{f_1, \ldots, f_m}$ of $m$ ($m \geq 2$) complex polynomials $f_1, \ldots, f_m$. We have proven that for $m \geq 3$ and for given vertex polynomials whose degrees differ at most by one, the Hurwitz stability of the entire polytope is equivalent to the Hurwitz stability of its exposed edges. Also, in case $m = 2$, we have proposed a necessary and sufficient condition for the Hurwitz stability of segments appointed by two complex polynomials of arbitrary degrees. The paper extends a few earlier results concerning the Hurwitz stability of polynomial families.

Recall that a necessary condition for a polytope of (real or complex) polynomials to be stable is that differences in degrees of vertex polynomials are not greater than 2 (see Lemma 1 in Białas and Góra [9]). In our opinion, an extension of Theorem 7 to polytopes of complex polynomials that can drop in degree by two at most is still an open problem.

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