Managing Value at Risk Using Put Options

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Abstract

A natural approach to reducing the risk of a position in stock, is by buying put options on the underlying. We consider a model where the Value at Risk is taken as measure of risk, in the framework of the Black-Scholes model. We show a method for the choice of the optimal strike price of the options, and provide an analytic formula for the optimal Value at Risk, for arbitrary hedging expenditure. We show that hedging with put options is more effective than hedging with forward contracts. We also investigate the impact of the position on Value at Risk if one is to terminate the investment prior to the maturity of the options. We demonstrate that American options do not provide better protection throughout the life span of the investment than European options.

Key Words: Value-at-Risk, risk management

1 Introduction

The Value at Risk (VaR), which is the worst case scenario of loss an investment might incur at a given confidence level, has established its position as one of the standard measures of risk, and is widely used throughout the field of finance and risk management. Even though VaR does have some undesirable properties due to its lack of coherence (see the work of Artzner, Delbaen, Eber and Heath [4]), the fact that in numerous cases it is straightforward and easy to compute has not diminished it’s popularity.

One of the most natural ideas to reduce the VaR of a position in stock is to buy European put options. By doing so one can cut off the undesirable scenarios, while leaving oneself open to the positive outcomes. A choice of a high strike price of the put option does cut off more of the unfavorable states, but at the same time produces higher hedging costs. The question of how to
balance the two trends so that the level of VaR is minimized was discussed by Ahn, Boudoukh, Richardson and Whitelaw [1, 2], under the assumptions of the Black-Scholes model. There it has been shown, that under the assumption that one considers a sufficiently small hedging expenditure, the optimal strike price of the put can be found. It also turned out that the strike price is independent of the level of hedging cost, and that the VaR under the optimal position has a closed-form analytic solution.

In recent years various generalizations of the problem turned in the direction of dynamic programming and utility maximization methods. For some of the examples see the work of Basak and Shapiro [5], Cuoco, He and Issaenko [6], and also Yiu [14]. There, instead of simply buying European put options, the investor engages in a self financing strategy, which maximizes the utility of terminal wealth, while keeping the level of VaR in check throughout the time of the investment. Such methods produce better results, though in many cases they require having to numerically solve PDEs or dynamic programming problems. A second drawback, with comparison to simple hedging with put options, is the fact that the investor constantly needs to rebalance the portfolio to follow the optimal strategy, which in practice is usually time consuming and costly. An advantage of hedging with put options that still remains, is the simplicity of the method.

In this paper we come back to the model of Ahn et al. We will provide an extension of the result, so that it allows for an arbitrary hedging expenditure. The extension was highlighted in [1], but no proof of the result has been given (The claim was only backed up by numerical evidence). We will show that if an investor has a single share and wishes to incur a cost exceeding the one necessary for the assumptions of Ahn et al. [1, 2], then he should choose the strike price of the put option so that its price is equal to the hedging expenditure. In other words, he should buy a single put option. The result seems quite natural if one imagines that the investor spends a "large" amount on hedging. Intuitively it is evident that it would not make much sense to buy more than one option per single share, since options are riskier than the underlying. On the other hand, buying less than one option would leave a part of the share exposed to risk, which when one spends enough on hedging looks to be a natural case to be avoided. The tools used for the proof of the result are quite elementary. The major obstacle that needs to be overcome is the ruling out of the case of over hedging. If one decides to buy more options than stock, then one does not have a closed form solution for the VaR. In order to optimize the strike price in such case it is therefore impossible to simply differentiate the VaR over the strike (as is done by Ahn [1, 2] under more stringent assumptions), since we have no formula for VaR.

Having found the optimal strike price, we will show that in the case of an arbitrary hedging expenditure we still have a simple analytic formula for the optimal VaR. We will show that allowing for arbitrary hedging expenditure significantly reduces the VaR in comparison with the constrained case. We will prove also that hedging with put options is more efficient than hedging with forward contracts or by investing in the risk free asset, and also demonstrate
how hedging with put options compares with optimal portfolios introduced by Emmer, Klüppelberg and Korn [10]. What is more, we will show that even though we hedge with European type options, the method also reduces the VaR if one is to terminate the investment prior to their maturity date. This follows from the fact that prior to maturity the put options can be sold at their market value, which turns out to have a desirable effect on the VaR of the position. From this perspective immunizing against risk using European options is not as static as it seems at first glance. It turns out that for the optimal choice of the strike price we also have an analytic formula for the VaR if we wish to terminate the investment at any time prior to the maturity of the option. Finally we show, that due to their lower prices, European puts are just as effective in immunizing as American put options.

The paper is organized as follows. In Section 2 we give preliminary results and notations. In Section 3 we first recall the main results of Ahn, Boudoukh, Richardson and Whitelaw [1, 2], then in Section 3.2 we turn to the generalization of the method to an arbitrary hedging expenditure. In Section 3.3 we compare the results of constrained and unconstrained hedging on a numeric example. In Section 3.4 we show that hedging with put options is an effective tool for the reduction of VaR. We prove that hedging with put options is more efficient than hedging with forward contracts and compare it with results attainable by investments into portfolios of several assets. Then in Section 4 we move to the computation of the VaR for investments terminated before the maturity of the option. In Section 4.2 we demonstrate that hedging with European options produces similar results to hedging with American puts. We finish off with conclusions in Section 5.

2 Preliminaries

In this section we introduce the necessary notations and give basic results, which will be used throughout the paper. We work under the assumptions of the Black-Scholes model. The behavior of the underlying asset $S_t$ is governed by the stochastic equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

(1)

where $\mu$ and $\sigma$ are the drift and the volatility, and $W_t$ is a Wiener process. The equation (1) has a closed form solution

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2}) t + \sigma W_t}.$$  

(2)

The Black-Scholes formula for the price of a European put option purchased at time $t$, with the exercise time $T$ and a strike price $K$, is given by the formula

$$P(t, K, S_t) = Ke^{-r(T-t)}N(-d_2(t, K, S_t)) - S_tN(-d_1(t, K, S_t)),$$  

(3)
where
\[ d_1(t, K, x) = \ln \left( \frac{x}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \frac{\sigma \sqrt{T - t}}{\sigma \sqrt{T - t}} \] (4)
\[ d_2(t, K, x) = d_1(t, K, x) - \sigma \sqrt{T - t}. \]

When the choice of \( t \) and \( S_t \) will be evident from the context we will sometimes simplify the above notations to \( P(K), d_1(K) \) and \( d_2(K) \).

Now we will turn to the definition of the value at risk. Let us first introduce some notations. We will assume that investments are made at time zero and are terminated at time \( t \). We will denote the value of a given portfolio at time \( t \) as \( V_t \). When computing a loss on a given investment we will take into account the time value of money, and therefore the present value of the loss at time \( t \) is computed as
\[ L = V_0 - e^{-rt}V_t. \] (5)

Throughout the paper we will use the value at risk as our risk measure.

**Definition 1** Value at risk at \( \alpha \) confidence level is the smallest number \( \text{VaR}_\alpha \), such that the probability that the loss of our portfolio exceeds this value is smaller than \( \alpha \). i.e.
\[ \text{VaR}_\alpha = \inf \{ Y \mid P(V_0 - e^{-rt}V_t > Y) < \alpha \}. \] (6)

In some cases the condition (6) can be simplified. The value at risk is a number \( \text{VaR}_\alpha \) which satisfies
\[ P(V_0 - e^{-rt}V_t > \text{VaR}_\alpha) = \alpha, \] (7)
if such a number exists. If not then the condition (6) needs to be applied.

Sometimes the computation of VaR for a given investment does not present a problem.

**Remark 2** If the loss of the portfolio can be expressed as \( f(Z) \), where \( f \) is a decreasing function and \( Z \) is a variable with a standard normal distribution, then VaR of such a portfolio is determined by the formula
\[ \text{VaR}_\alpha = f(c(\alpha)), \] (8)
where \( c(\alpha) \) is the cut off point for the standard normal distribution at \( \alpha \) (i.e. \( P(Z < c(\alpha)) = \alpha \)).

### 3 VaR of a position in stock hedged by an investment in put options

In the first part of the section we are going to take a brief look at the results of Ahn, Boudoukh, Richardson and Whitelaw [1, 2], where the problem of hedging
the value at risk of a position in a single share is considered. The hedging strategy involves buying European put options for a fixed amount $C$. The question is how to find the optimal strike price of the put options which we should use in order to perform our hedge, so that the VaR computed for the expiry date $T$ of the options is as small as possible. Let us note that for the result to hold it is important that the amount $C$ is small in comparison to the stock price $S_0$. The second task of the section is to investigate what happens when the $C$ is relatively large. In the third part we are going to compare the two results on a numerical example.

3.1 The case of small hedging expenditure

In this section we recall the results of Ahn et al. The following Theorem is the main tool used for finding the optimal strike price.

**Theorem 3** The value at risk at $\alpha$ confidence level of a position consisting of a single share and $h \leq 1$ put options with a strike price $K$ and an expiry date $T$ is equal to

$$ VaR_\alpha(K) = S_0 + hP(K) - e^{-rT}S_T(c(\alpha)) - e^{-rT}h(S_T(c(\alpha)) - K) \quad (9) $$

where $c(\alpha) \in \mathbb{R}$ is the cut off point for the standard normal distribution at $\alpha$ (i.e. $P(Z < c(\alpha)) = \alpha$) and $S_T(c(\alpha)) = S_0e^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}c(\alpha)}$.

We can see from the formula (9) that if we chose $K$ smaller than $S_T(c(\alpha))$, then the value at risk of our position is higher than the value at risk of a single position in stock, given by

$$ VaR_\alpha = S_0 - e^{-rT}S_T(c(\alpha)). \quad (10) $$

Buying a put option with such a $K$ does not decrease VaR of our position. This makes sense since there is no point in buying a put option with a strike price which is so low, that the event of ever having to use it is very unlikely. Such an option will not protect us from the loss at $\alpha$ confidence level.

Let us note that if we intend to use a fixed amount $C$ for hedging our position with put options, we have the following constraint

$$ C = hP(0, K, S_0). $$

From this formula we can see that $h$ is not fixed, but depends on the choice of the strike price $K$.

The formula (9) is used to compute the optimal strike price $K_0$ of a put option, which we should buy in order to minimize the VaR of our position. To find the optimal $K_0$ we simply compute the derivative of (9) with respect to $K$ and equate it to zero

$$ \frac{d}{dK} \left( S_0 + C - e^{-rT}S_T(c(\alpha)) - e^{-rT} \frac{C}{P(K)}(K - S_T(c(\alpha))) \right) = 0. \quad (11) $$
The above derivative is equal zero for \( K = K_0 \) which satisfies the following condition
\[
S_T(c(\alpha)) = S_0 \frac{N(-d_1(0, K_0, S_0))}{e^{-rT}N(-d_2(0, K_0, S_0))}
\] (12)

Even though the expression (12) does not provide an analytical formula for the optimal \( K_0 \), it can be used to solve the problem numerically.

3.2 The case of arbitrary hedging expenditure

Let us now consider the case when it turns out that the amount used for hedging \( C \) exceeds the price of the put option \( P(K_0) \), where \( K_0 \) is the optimal strike price suggested by (12). In such a case we can no longer rely on the choice of the strike price \( K_0 \), since then \( h = C/P(K_0) \) is greater than one and the result of the Theorem 3 no longer holds. Thus (12) does no longer refer to the minimization of the VaR.

To state our result let us first introduce some notations. Let \( K^* \) be the strike price such that the price of the put option is equal to \( C \)
\[
P(K^*) = C.
\] (13)

Let us also denote by \( Y^* \) the value at risk of a portfolio consisting of a single stock and a single put with the strike price \( K^* \). Since in this case \( h = 1 \) we can apply (9) and compute \( Y^* \) analytically
\[
Y^* = \text{VaR}_\alpha(K^*) = S_0 + P(K^*) - e^{-rT}S_T(c(\alpha)) - e^{-rT}(K^* - S_T(c(\alpha)))_+.
\] (14)

Using the above notation we have the following generalization of Theorem 3.

**Theorem 4** If we buy a single share and wish to spend an amount \( C \) on hedging our position by buying put options, then we should buy put options with the strike price
\[
K_{opt} = \begin{cases} K_0 & \text{if } \frac{C}{P(K_0)} \leq 1 \\ K^* & \text{if } \frac{C}{P(K_0)} > 1 \end{cases}
\] (15)

where \( K_0 \) and \( K^* \) is given by (12) and (13) respectively. The value at risk of our position at \( \alpha \) confidence level is
\[
\text{VaR}_\alpha(C) = \text{VaR}_\alpha(K_{opt})
\]
\[
= S_0 + C - e^{-rT}S_T(c(\alpha)) - e^{-rT}\frac{C}{P(K_{opt})}(K_{opt} - S_T(c(\alpha)))_+.
\] (16)

**Remark 5** The above principle (15) has been already indicated in [1], but due to the fact that there is no analytic formula for \( \text{VaR} \) in case of \( h > 1 \), the conjecture has not been proved (It has been only backed up by numerical examples). The main difficulty in the proof of Theorem 4, is to obtain the result without a formula for \( \text{VaR} \).
Figure 1: The loss function $L$ for the portfolio consisting of a single stock and $h > 1$ put options.

**Proof of Theorem 4.** The first case in (15) follows from Theorem 3. First we will consider a situation when we choose $C$ and $K$ such that $C e^{rT} P(K) > 1$ and $K < K^*$. We will show that for such $K$ the value at risk of the investment is greater than $Y^*$. Let us first consider the loss function $L_K$ with respect to the future value of stock

$$L_K(S_T) = S_0 + C - e^{-rT} S_T - e^{-rT} C P(K) (K - S_T)_+.$$  \hspace{1cm} (17)

If we choose a certain loss level $Y$, then it is not difficult to compute the probability that the loss will exceed this given value. In order to do this we can first compute the values $x_1$ and $x_2$ (see Figure 1)

$$x_1(Y) = \max \left( \frac{e^{rT} (Y - S_0 - C) + C P(K) K}{P(K)} - 1, 0 \right), \hspace{1cm} (18)$$

$$x_2(Y) = e^{rT} (S_0 + C - Y). \hspace{1cm}$$

To find our probability $P(L_K > Y)$ all we need to compute now, is the probability that $S_T$ is in the interval $(x_1(Y), x_2(Y))$. Since $S_T$ is given by (2) this probability is equal to

$$P(L_K > Y) = P(S_T \in (x_1(Y), x_2(Y))) = N(g(x_2(Y))) - N(g(x_1(Y))), \hspace{1cm} (19)$$

where

$$g(x) = \ln \left( \frac{x}{S_0} \right) - \left( \mu - \frac{\sigma^2}{2} \right) T \sigma \sqrt{T}. \hspace{1cm} (20)$$

Let us now define a function $q(K)$ as the probability that the loss $L_K$ is greater than $Y^*$

$$q(K) = P(L_K > Y^*). \hspace{1cm}$$
Let us show that the function \( q(K) \) is a decreasing function on the interval \((0, K^*)\). We have

\[
\frac{d}{dK} q(K) = \frac{d}{dK} \left[ N(g(x_2(Y^*))) - N(g(x_1(Y^*))) \right]
= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{g(x_2(Y^*))^2}{2}\right) \frac{dg}{dx}(x_2(Y^*)) \frac{d}{dK} x_2(Y^*)
- \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{g(x_1(Y^*))^2}{2}\right) \frac{dg}{dx}(x_1(Y^*)) \frac{d}{dK} x_1(Y^*).
\]

We note that \( x_2 \) is independent from \( K \), which means that the first term in our equation is equal to zero. It is easy to check that \( \frac{dg}{dx} > 0 \), which means that in order to show that \( \frac{d}{dK} q(K) < 0 \) it is enough to show that

\[
\frac{d}{dK} x_1(Y^*) > 0.
\]

We have

\[
\frac{d}{dK} x_1(Y^*) = \frac{d}{dK} \left( \frac{e^{rT}(Y^* - S_0 - C) + C}{P(K) - 1} \right)
= \frac{d}{dK} \left( -e^{rT}(Y^* - S_0 - C) + C \frac{e^{rT}(Y^* - S_0 - C) + K}{C - P(K)} \right)
= C \frac{C - P(K) + \frac{dP}{dK} e^{rT}(Y^* - S_0 - C) + K}{(C - P(K))^2}.
\]

Since we are only interested whether the above expression is positive, we can drop the term \( \frac{C}{(C - P(K))^2} \). Using the formulas (3) and the fact that \( \frac{dP}{dK} = e^{-rT} N(-d_2(K)) \), we obtain

\[
C - P(K) + \frac{dP}{dK} (e^{rT}(Y^* - S_0 - C) + K)
= C - K e^{-rT} N(-d_2(K)) + S_0 N(-d_1(K))
+ e^{-rT} N(-d_2(K)) (e^{rT}(Y^* - S_0 - C) + K)
= C + S_0 N(-d_1(K)) + N(-d_2(K))(Y^* - S_0 - C).
\]

Using (14) and (13) we have

\[
C + S_0 N(-d_1(K)) + N(-d_2(K))(Y^* - S_0 - C)
= P(K^*) + S_0 N(-d_1(K)) - e^{-rT} K^* N(-d_2(K))
= P(K^*) + S_0 N(-d_1(K)) - e^{-rT} K N(-d_2(K))
- e^{-rT} (K^* - K) N(-d_2(K))
= P(K^*) - P(K) - (K^* - K) \frac{dP}{dK} (K)
> 0.
\]
Figure 2: The loss function $L_K$ and VaR$_{\alpha}$ for $K < K^*$. 

The last inequality is a result of the fact that the price function $P(K)$ is convex with respect to $K$.

Since $q(K) = P(L_K > Y^*)$ is a decreasing function, for $K < K^*$ we have $P(L_K > Y^*) > P(L_{K^*} > Y^*) = \alpha$. This and the fact that $P(L_K > VaR_{\alpha}(K)) = \alpha$, gives us $VaR_{\alpha}(K) > Y^* = VaR_{\alpha}(K^*)$, for all $K < K^*$ (see Figure 2).

We have so far shown that if we consider the strike prices equal to or smaller than $K^*$, then the smallest value at risk for our portfolio is obtained by spending $C$ on a single put option with the strike price $K^*$. What is left now is to show that if we choose $K > K^*$ then $VaR_{\alpha}(K)$ will be greater than $VaR_{\alpha}(K^*)$. In this case the proportion $h$ of the put options, which we can buy for our amount $C$ is smaller than one

$$h = \frac{C}{P(K)} < \frac{C}{P(K^*)} = 1,$$

and therefore to compute the value at risk we can apply the formula (9)

$$VaR_{\alpha}(K) = S_0 + C - e^{-rT}S_T(c(\alpha)) - e^{-rT}\frac{C}{P(K)}(K - S_T(c(\alpha)))_+.$$  \hspace{3cm} (21)

We know that this function takes the smallest value at $K_0$ and that it is increasing on the interval $[K_0, +\infty)$. We also have $K^* > K_0$. We want to find the smallest value of $VaR_{\alpha}(K)$ on the interval $[K^*, +\infty)$. It is clear that this is attained at $K = K^*$. This finishes our proof. \hfill \blacksquare

As a finishing remark for this section, let us notice that spending a large amount $C$ on the hedge, gives us the ability to reduce the value at risk of our position arbitrarily close to zero.

Remark 6 The value at risk can be reduced arbitrarily close to zero

$$\lim_{C \to \infty} VaR_{\alpha}(C) = 0.$$
Proof. From (16), (3) and (4) we can compute

\[
\lim_{C \to \infty} \text{Var}_\alpha(C) = \lim_{C \to \infty} [S_0 + C - e^{-rT} S_T(c(\alpha)) - e^{-rT}(K^*(C) - S_T(c(\alpha)))_+]
\]

\[
= \lim_{C \to \infty} [S_0 + C - e^{-rT} K^*(C)]
\]

\[
= \lim_{K \to \infty} [S_0 + P(K) - e^{-rT} K]
\]

\[
= \lim_{K \to \infty} [S_0(1 - N(-d_1(K))) + e^{-rT} K(N(-d_2(K)) - 1)]
\]

\[
= e^{-rT} \lim_{K \to \infty} [K(N(-d_2(K)) - 1)]
\]

\[
= 0.
\]

This fact does not mean that we are decreasing the VaR of our position for free. This is reflected by the expected rate of return on our investment for high values of \( C \). The higher we choose \( C \), the closer the return gets to the risk free rate \( r \) and therefore as \( C \) increases, the investment becomes less and less profitable.

3.3 Comparison of the results with limited and unlimited hedging expenditure

In this section we are going to present some numerical results. We will follow the initial data considered by Ahn, Boudoukh, Richardson and Whitelaw [2], and examine what impact the extension onto arbitrary hedging expenditure has on VaR reduction. We will also demonstrate the dependence of the VaR on the choice of the strike price.

The following parameters were considered: \( S_0 = 100, \mu = 0.10, \sigma = 0.15, r = 0.05, T = 1 \) and \( \alpha = 2.5\% \). From (10) we find that the VaR of an unhedged position in a single share is equal to 22.53. If one wishes to hedge the position by buying European put options then from (12) the optimal strike price \( K_0 \) is equal to 87.59. The maximal hedging expenditure for which the results of Ahn et al. hold (i.e. for which \( h = C/P(K_0) \leq 1 \)) is \( \bar{C} = 0.74 \). With such maximal hedging, using (9), we find that the VaR is reduced to 17.43. If we choose to hedge with \( C > \bar{C} \) then \( K_0 \) is no longer the optimal choice. In such case we have to choose the strike price \( K_{\text{opt}} \) such that \( P(K_{\text{opt}}) = C \). The choice of the strike
Figure 3: The optimal strike price $K_{opt}$ for various levels of hedging expenditure $C$.

The optimal strike price for various hedging expenditures is shown in Table 1. We can see that for $C$ up to $\bar{C}$ (depicted in bold font) we have a constant optimal strike price. Above $\bar{C}$ the optimal strike price depends on the choice of $C$. We can see that choosing higher $C$ significantly reduces the VaR of our position. These results are also graphically depicted in Figures 3 and 4, where the black dot represents the result for $\bar{C}$.

<table>
<thead>
<tr>
<th>$C$</th>
<th>$K_{opt}$</th>
<th>$VaR_\alpha(K_{opt})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>87.59</td>
<td>21.15</td>
</tr>
<tr>
<td>0.50</td>
<td>87.59</td>
<td>19.08</td>
</tr>
<tr>
<td><strong>0.74</strong></td>
<td><strong>87.59</strong></td>
<td><strong>17.43</strong></td>
</tr>
<tr>
<td>0.75</td>
<td>87.67</td>
<td>17.36</td>
</tr>
<tr>
<td>1.00</td>
<td>89.50</td>
<td>15.86</td>
</tr>
<tr>
<td>1.50</td>
<td>92.32</td>
<td>13.68</td>
</tr>
</tbody>
</table>

Table 1: The optimal strike price $K_{opt}$ and the resulting optimal $VaR_\alpha$ for various hedging expenditures $C$.

Let us now briefly discuss how the VaR of our position depends on the choice of the strike price of the put option. The results are given in Figure 5. For $C < \bar{C}$ we can use (9) to compute $VaR_\alpha(K)$. An example of this case is for $C = 0.5$. For $C > \bar{C}$ the formula (9) holds only for $K$ above $K_{opt}$, and for smaller $K$ one has to use (19) to find $VaR_\alpha(K)$. The kink in the graph at the minimum is the result of the fact that the formula (9) brakes down for $K < K_{opt}$, since for such $K$ the fraction $h = C/P(K)$ invested in the put options is greater than one. In
Figure 4: The optimal $VaR_{\alpha}$ for various hedging expenditure $C$.

Figure 5 we draw the results from $C = 0.75$, which is just above the boundary $\bar{C} = 0.74$, up to $C = 1.5$. For higher $C$ the kink in the graph is more evident. Also the minimal attainable VaR is clearly lower than for small $C$.

### 3.4 VaR / expected income analysis

A natural question to ask is whether hedging with put options is justified. At first glance one might argue that options are relatively expensive and therefore would probably not be very efficient. The simplest, cheapest and most natural idea for hedging of a position in stock is signing a forward contract on it (or on a part of it), or investing part of the money risk free. In this section we will show that put options perform far better. We will also compare how hedging with put options compares with diversifying the investment into various assets.

Let us start with considering what will happen when we sign a forward contract on a portion $\lambda \in [0,1]$ of a single unit of stock. We have $V_0^F = S_0$, $V_T^F(\lambda) = (1 - \lambda)S_T + \lambda S_0 e^{rT}$ and the loss for our investment will be

$$L = S_0 - e^{-rT} ((1 - \lambda)S_T + \lambda S_0 e^{rT}) .$$

This means that we can apply Remark 2 to obtain

$$VaR_\alpha^F(\lambda) = (1 - \lambda) \left( S_0 - e^{-rT} S_T(c(\alpha)) \right)$$

(23)

(Note also that investing $(1 - \lambda)S_0$ in stock and $\lambda S_0$ risk free will give us an identical result to (23)).

If on the other hand we invest $(1 - \lambda)S_0$ into stock and spend $\lambda S_0$ on hedging
Figure 5: The dependence of \( \text{VaR}_\alpha \) on the choice of the strike price \( K \) when the hedging expenditure is fixed at various levels.

with put options then applying Theorem 4 gives us

\[
\text{VaR}_{\alpha}^{\text{put}}(\lambda) = S_0 - e^{-rT} \left((1 - \lambda)S_T(c(\alpha)) + \frac{\lambda S_0}{P(K_{opt})} (K_{opt} - S_T(c(\alpha)))_+ \right),
\]

(24)

with \( K_{opt} = \max(K_0, K^*) \) and

\[
P(K^*) = \frac{\lambda S_0}{1 - \lambda}.
\]

(25)

**Proposition 7** For any hedging fraction \( \lambda \in (0, 1) \) we have

\[
\text{VaR}_{\alpha}^{\text{put}}(\lambda) < \text{VaR}_{\alpha}^{\text{F}}(\lambda).
\]

**Proof.** From (22) and (23) we know that

\[
\text{VaR}_{\alpha}^{\text{F}}(1) = \text{VaR}_{\alpha}^{\text{put}}(1) = 0.
\]

(26)

Let \( \lambda^* \) be such that \( \frac{\lambda^* S_0}{1 - \lambda^*} = K_0 \). For \( \lambda \in [\lambda^*, 1) \) using (23), (24), (25) we have

\[
\text{VaR}_{\alpha}^{\text{F}}(\lambda) - \text{VaR}_{\alpha}^{\text{put}}(\lambda) = e^{-rT} \left((1 - \lambda)(K^* - S_T(c(\alpha))) - \lambda S_0 e^{rT} \right).
\]

The fact that \( K^* = P^{-1}(\frac{\lambda S_0}{1 - \lambda}) \) and \( \frac{d}{dK} P(K) = e^{-rT} N(-d_2) \) gives

\[
\frac{\partial}{\partial \lambda} \left( \text{VaR}_{\alpha}^{\text{F}}(\lambda) - \text{VaR}_{\alpha}^{\text{put}}(\lambda) \right) =
\]

\[
= e^{-rT} \left( -(K^* - S_T(c(\alpha))) + \frac{1}{e^{-rT} N(-d_2)} \frac{S_0}{(1 - \lambda)} - S_0 e^{rT} \right)
\]

\[
< \frac{S_0}{(1 - \lambda)} - S_0
\]

< 0.
which together with (26) gives \( \text{VaR}^F_{\alpha}(\lambda) > \text{VaR}^\text{put}_{\alpha}(\lambda) \) for \( \lambda \in [\lambda^*, 1) \).

For \( \lambda \in (0, \lambda^*) \) we have \( K_{\text{opt}} = K_0 \) and

\[
\text{VaR}^F_{\alpha}(\lambda) - \text{VaR}^\text{put}_{\alpha}(\lambda) = \lambda S_0 \left( \frac{K_0 - S_T(c(\alpha))}{e^{rT} P(K_0)} - 1 \right).
\]

Using (12) gives us

\[
e^{rT} P(K_0) = K_0 N(-d_2) - S_0 e^{rT} N(-d_1) = K_0 N(-d_2) - S_T(c(\alpha)) N(-d_2) < K_0 - S_T(c(\alpha)),
\]

hence \( \text{VaR}^F_{\alpha}(\lambda) > \text{VaR}^\text{put}_{\alpha}(\lambda) \) for \( \lambda \in (0, \lambda^*) \).

We can continue the example from Section 3.3 to compare the results obtained with (23) and (24). It turns out (see Figure 6) that hedging with put options is by far more effective. In comparison with buying bonds or signing forward contracts, buying put options requires far less hedging expenditure and also has an immediate effect even with a very small fraction spent on hedging.

One might wonder whether the VaR reduction performed with put options does not significantly reduce the expected proceeds. To answer this question let us note that

\[
E(V^F_T(\lambda)) = (1 - \lambda) S_0 e^{\mu T} + \lambda S_0 e^{rT}.
\]

Using the fact that

\[
E(K - S_T)_+ = e^{\mu T} P_\mu(K) = KN(-d_2^\mu) - e^{\mu T} S_0 N(-d_1^\mu)
\]

(where \( d_i^\mu \) is computed identically to \( d_i \), but with \( \mu \) substituted instead of \( r \)) we have

\[
E(V^\text{put}_T(\lambda)) = (1 - \lambda) S_0 e^{\mu T} + \frac{\lambda S_0}{P(K_{\text{opt}})} e^{\mu T} P_\mu(K).
\]
Definition 8 We say that an investment with expected income $E_1$ and value at risk $VaR_1$ dominates an investment with $E_2$ and $VaR_2$ if 

$$E_1 \geq E_2 \quad \text{and} \quad VaR_1 \leq VaR_2.$$

Proposition 9 If $\mu > r$ then investments obtained by hedging with put options dominate investments obtained by hedging with forwards.

Proof. We start by observing that 

$$(VaR^\text{put}_\lambda, E(V^\text{put}_T(\lambda))) = (VaR^\text{F}_\lambda, E(V^\text{F}_T(\lambda))) \quad \text{for } \lambda \in \{0, 1\}.$$ 

Since $\mu > r$ we have $e^{\mu T}P_\mu(K) < e^{r T}P(K)$. For $\lambda \in (0, 1)$, using (27), (28), Proposition 7 and (26) this gives us 

$$\frac{E^\text{put}(\lambda) - E^\text{put}(1)}{VaR^\text{put}(\lambda) - VaR^\text{put}(1)} = \frac{(1 - \lambda)S_0e^{\mu T} - \frac{(1-\lambda)S_0}{P_\lambda(K_{opt})} e^{\mu T}P_\mu(K_{opt})}{VaR^\text{put}(\lambda)}$$ 

$$> \frac{(1 - \lambda)S_0e^{\mu T} - (1 - \lambda)S_0e^{r T}}{VaR^\text{F}(\lambda)}$$ 

$$= \frac{E^\text{F}(\lambda) - E^\text{F}(1)}{VaR^\text{F}(\lambda) - VaR^\text{F}(1)}.$$ 

Continuing our example, combining (23) with (27) and also (24) with (28) allows us to plot the results on the VaR / expected income plane (see Figure 7). Looking at the graph we can see that portfolios constructed with the use of put options dominate the ones attainable with forward contracts. Hedging with put options clearly outperforms hedging with forwards.
We should now ask whether our hedging with put options is not in some sense artificial. From Figure 6 we can see that the VaR reduction is substantial, but we do not see what strike prices are required for particular parameters $\lambda$. If it turned out that the required strike prices are too high, then our use of the Black-Scholes model would be unjustified since we would be in the fat-tail region of the true future stock price distribution. Looking at Figure 8 we can see that the required strike prices for $\lambda \in [0, 0.3]$ remain within a reasonable range. For $\lambda \in [0, 0.1]$ the required strike prices certainly seem quite natural. For these parameters the use of the Black-Scholes model should be a reasonable approximation. In any case the nature of Figure 6 should remain the same for more general (say with stochastic volatility or jump diffusion) setting.

Another method of optimizing risk is by diversification. We will now compare how the reduction of VaR with put options compares with the optimal portfolio selection developed by Emmer, Klüppelberg and Korn [10]. There the following situation is considered. We have one risk free and $d$ risky assets with the following dynamics

$$
\begin{align*}
    dS_0^t &= S_0^t \dt, & S_0^0 &= 1, \\
    dS_i^t &= S_i^t \left( \mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW^j_t \right) \quad \text{for } i = 1, \ldots, d.
\end{align*}
$$

Where $W_t = (W^1_t, \ldots, W^d_t)$ is a $d$-dimensional Wiener process. We will use notations $\sigma = (\sigma_{ij})_{i,j=1,\ldots,d}$, $m = (\mu_1, \ldots, \mu_d)'$, $1 = (1, \ldots, 1)'$. We consider portfolios $\pi = (\pi_1, \ldots, \pi_d)$ where $\pi_i$ is a fraction (constant in time) of the initial wealth which is invested into an asset $i$. The wealth at time $t$ of the portfolio is

$$
V_\pi^t = (1 - \sum_{i=1}^{d} \pi_i) S_0^t + \sum_{i=1}^{d} \pi_i S_i^t.
$$

The initial wealth is $x$ and the following problem investigated

$$
\max E(V_\pi^T) \quad \text{subject to } \text{VaR}_\alpha(V_\pi^T) \leq C \tag{29}
$$

**Theorem 10** [10] Let $\theta = ||\sigma^{-1}(m - r1)||$ and assume that $\mu_i \neq r$ for at least one $i \in \{1, \ldots, d\}$. Assume furthermore that $C$ satisfies

$$
\begin{align*}
    0 &\leq C \leq xe^{\theta T} \quad \text{if } \theta \sqrt{T} < c(\alpha), \\
    xe^{\theta T} \left( 1 - e^{\left( \frac{1}{2}(\sqrt{T}\theta) - |c(\alpha)| \right)^2} \right) &\leq C \leq xe^{\theta T} \quad \text{if } \theta \sqrt{T} \geq c(\alpha).
\end{align*}
$$

Then problem (29) is solved by

$$
\pi^* = \frac{(\sigma \sigma')^{-1}(m - r1)}{||\sigma^{-1}(m - r1)||}
$$

with

$$
\epsilon^* = \theta + \frac{c(\alpha)}{\sqrt{T}} + \sqrt{\left( \theta + \frac{c(\alpha)}{\sqrt{T}} \right)^2 - \frac{2c}{T}},
$$

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where \( c = \ln \left( 1 - \frac{\xi}{\bar{x}} \right) \). The corresponding maximal expected terminal wealth under the VaR constraint is
\[
E(V_T^{\pi^*}) = xe^{(r+\varepsilon^*\theta)T}.
\]

In our case \( x = S_0 \), and using the above Theorem we can easily obtain a curve of VaR optimal portfolios obtained by investments into a single share and a risk free asset. The plot of this curve (for our numerical example) on the VaR - expected income and on the \( \lambda \)- VaR plane is given in Figures 7 and 6 respectively. It turns out that such portfolios are outperformed by both simple hedging with forwards and by hedging with put options. Let us note that such portfolios also require constant change in time of the investment strategy in order to keep the proportions invested in stock and risk free asset constant. From this perspective we can see that portfolios which only consider single stock and the risk free asset are clearly undesirable.

The nature of Theorem 10 though is not to consider just a single asset. To exploit it one should consider at least several stocks combined with a risk free asset. Depending on the matrix \( \sigma \) the results of Theorem 10 in comparison to hedging with put options vary. In Figure 7 we have considered three uncorrelated stocks with identical coefficients \( \mu_i = 0.1, \sigma_i = 0.15 \) for \( i = 1, 2, 3 \) (we take \( \sigma_{ij} = 0 \) for \( i \neq j \)) together with a risk free asset. In such setting in some cases hedging with put options will outperform optimal portfolios. An example of this would be for \( VaR = 5 \) (see Figure 7).

This by no means is a typical picture. The optimal portfolios strongly depend on \( \sigma \). Taking for example \( \sigma_{12} = \sigma_{21} = -0.01 \) and keeping the rest of parameters we obtain optimal portfolios as in Figure 9. In fact this is a more typical scenario than Figure 7 and in many cases optimal portfolios will outperform simple...
hedging with put options on a single stock. Taking into consideration Figure 7 though, we can see that put options can improve the position. A natural question to address at this point is the problem of Markowitz type return-VaR optimization (in the spirit of Emmer, Klüppelberg and Korn [10], or Alexander and Baptista [3]), which would also include put options. The answer to this is not straightforward and will be the focus of forthcoming work.

Even though hedging Value at Risk with portfolios is in many cases more efficient, in the case when we have a given position in a single type of stock, hedging with portfolios might not be most favourable. This is due to the fact that we would need to take a given position on different types of stocks. For optimal portfolios, most often the position in each type of stock is just a fraction of the overall investment. This would mean that we either have to buy a substantial amount of new stocks (most likely exceeding in value the value of stock which we want to hedge) or sell our stock and rebalance our position. This in many cases might be undesirable. Hedging with put options in such a case seems to be an attractive alternative since it does not require a large capital input.

4 Hedging of VaR for times smaller than the maturity of the option

The aim of this section is to show that taking a position in stock and put options not only reduces the value at risk of the position at the terminal date of the options, but also immunizes if the investor is to terminate the position prior to the maturity. A natural idea when one thinks of reducing the risk throughout the entire time interval of an investment, is to do so using American put options. We will demonstrate that immunization using European put options produces better
results than with American options. We will adopt the following approach. We will assume that the investor chooses his position in order to minimize the VaR at the terminal date. As a by-product the position is also immunized for earlier dates. We will show how this immunization compares, when hedging with European and American options.

4.1 Hedging with European put options

If we buy a European put option with a maturity date \( T > 0 \), then this reduces the Value at Risk of a position in stock, if we were to terminate our investment at an earlier date than the maturity. This follows from the fact, that even though the options considered are European, we can always sell them at their market price. In the case when we decide to buy less options than stock the result turns out to be quite elementary.

**Proposition 11** Let us consider a position in a single stock and \( h = C/P(K) \leq 1 \) European put options with the strike price \( K \) and maturity date \( T \). Then the value at risk at the date \( t \) of the position is equal to

\[
VaR_{\alpha,t}(K) = S_0 + C - e^{-rt}S_t(c(\alpha)) - e^{-rt}\frac{C}{P(K)}P(t, K, S_t(c(\alpha))),
\]

where \( P(t, K, x) \) is the time \( t \) price of the put option given by (3).

**Proof.** Let us consider the loss function for the investment computed at the time \( t \)

\[
L_K(S_t) = S_0 + C - e^{-rt}S_t - e^{-rt}\frac{C}{P(K)}P(t, K, S_t).
\]

Computing the partial derivative with respect to \( x = S_t \) and using the fact that \( N(\cdot) < 1 \) and \( \frac{C}{P(K)} \leq 1 \) we obtain

\[
\frac{\partial L_K}{\partial x} = -e^{-rt}\left(1 + \frac{C}{P(K)}\frac{\partial}{\partial x}P(t, K, x)\right)
\]

\[
= -e^{-rt}\left(1 - \frac{C}{P(K)}N(-d_1(t, K, x))\right)
\]

\[
< 0.
\]

This means that the function \( L_K(x) \) is decreasing. Defining \( f(Z) = L_K(S_t(Z)) \) where \( S_t(Z) \) is given by (2) and applying Remark 2 we obtain our claim.

In this section we work under the assumption that the investor chooses the strike price so so that the value at risk is minimized at the terminal date. In this case the Proposition 11 proves a sufficient result, since from Theorem 4 we have \( C/P(K_{opt}) \leq 1 \), regardless of the hedging expenditure level.

**Remark 12** Let us note that the optimal choice of the strike price for the terminal date, is not optimal for all the earlier times. It turns out that for different
times, different strike prices optimize the value at risk. A global approach to minimization over the entire time interval might prove problematic. This is due to the fact, that for some earlier dates, the optimal strike price surprisingly results in a position in which we have more puts than stock. These cases are not tractable analytically. Since they also defy economic common sense we will not enter into the discussion.

Now let us turn to a numerical example. We follow the example from the Section 3.3, using the same initial data as before. Using Proposition 11 we can compute the value at risk of the position in stock and European put options. We choose a given hedging expenditure $C$ from zero up to one, and examine $\text{VaR}_{\alpha,t}(K_{\text{opt}})$ using (30). The results are given in Table 2 and Figure 10. From the results we can see that even though we were aiming to hedge our position at the expiry date $T = 1$, by doing so we are also reducing $\text{VaR}$ for times $t < T$.

$$
\begin{array}{cccc}
 t & C = 0 & C = 0.5 & C = 1 \\
0.2 & 11.64 & 10.45 & 9.43 \\
0.4 & 15.67 & 13.88 & 12.33 \\
0.6 & 18.49 & 16.22 & 14.20 \\
0.8 & 20.70 & 18.00 & 15.45 \\
1.0 & 22.53 & 19.08 & 15.86 \\
\end{array}
$$

Table 2: The $\text{VaR}_{\alpha,t}(K_{\text{opt}})$ for dates $t$ prior to the expiry of the option for various choices of the hedging expenditure.

4.2 Comparison of the results for hedging with European and American options

We will now turn to the hedging with American options. Let $P_A^t(K, S_t)$ denote the time $t$ price of an American option with the terminal date $T$, the price of the underlying stock $S_t$ and strike price $K$. If an investor decides to hedge the position in single stock by buying $C/P_A^0(K, S_0)$, then if he terminates the investment at time $t \leq T$ then he can either realize or sell the put. Since $P_A^t(K, S_t)$ is decreasing with respect to $S_t$, using the fact that $P_A^t(K, S_t) \geq (K - S_t)_+$, the resulting value at risk of the investment, terminated at time $t \leq T$, has a similar form to (9)

$$
\text{VaR}_{\alpha,t}(K) = S_0 + C - e^{-rt}S_t(c(\alpha)) - e^{-rt} \frac{C}{P_A^0(K)} P_A^t(K, S_t(c(\alpha))).
$$

(32)

In the case of the American options simple analytic optimization of the value at risk using (32) is impossible, since we have no formula for $P_A^t(K, S_t)$. The only
Figure 10: The VaR, when hedging with European options, for the optimal choice of the strike price \( K_{opt} \) and for times \( t \) prior to the maturity \( T \).

Attainable investigation is numerical, and an example of such is given later on in the section.

A tractable case for the value at risk optimization can be made, when we investigate a perpetual put instead of a standard American put option. In this case we have the following formula for the price of the option (see for example Elliott and Kopp [9] for more details on the perpetual put)

\[
P_A^P(K, S_t) = \begin{cases} 
\left( \frac{K}{1+\gamma} \right)^{1+\gamma} \left( \frac{S_t}{\gamma} \right)^{-\gamma} & \text{if } K < K^* \\
K - S_t & \text{if } K \geq K^*,
\end{cases}
\]

(33)

where \( K^* = K^*(S_t) = \frac{S_t(1+\gamma)}{\gamma} \) and \( \gamma = \frac{2\tau}{\sigma^2} \). We will work under an assumption that \( S_T(c(\alpha)) < S_0 \), which is natural when working with real life (or just reasonable) data. We have the following result.

**Proposition 13** If we wish to immunize our position at time \( T > 0 \) against VaR by buying perpetual put options, then if we invest \( C \leq \frac{S_T(c(\alpha))^{1+\gamma}}{S_0} \) into hedging, then it is optimal to buy put options with a strike price \( S_T(c(\alpha))^{1+\gamma} \) or smaller. The resulting optimal VaR is

\[
VaR_{opt} = S_0 + C - e^{-rT} S_T(c(\alpha)) - e^{-rT} C \left( e^{\left( \mu - \frac{\sigma^2}{2} \right)T + \sigma \sqrt{T} c(\alpha)} \right)^{-\gamma}.
\]

(34)

What is more, if we were to terminate the investment at time \( t < T \), then the resulting VaR is

\[
VaR_t = S_0 + C - e^{-rt} S_t(c(\alpha)) - \frac{e^{-rt} C \gamma (S_T(c(\alpha))^{1+\gamma} - S_t(c(\alpha)))_+}{S_T(c(\alpha)) (1 + \gamma)}.
\]

(35)
\textbf{Proof.} If we buy perpetual put options with a strike price \( K \leq K^\ast(S_T(c(\alpha))) \) then from (32) and (33) we have

\begin{equation}
\text{VaR}_{\alpha,T}(K) = S_0 + C - e^{-rT}S_T(c(\alpha)) - e^{-rT}C \left( \frac{S_T(c(\alpha))}{S_0} \right)^{-\gamma}.
\end{equation}

When \( K \in [K^\ast(S_T(c(\alpha))), K^\ast(S_0)] \) then

\begin{equation}
\text{VaR}_{\alpha,T}(K) = S_0 + C - e^{-rT}S_T(c(\alpha)) - e^{-rT}C \left( \frac{K - S_T(c(\alpha))}{S_0} \right)^{1+\gamma} \left( \frac{S_0}{S_T(c(\alpha))} \right)^{-\gamma}
\end{equation}

and \( \frac{d}{dK} \text{VaR}_{\alpha,T}(K) = 0 \) for \( K = K^\ast(S_T(c(\alpha))) \). The case of \( K > K^\ast(S_0) \) does not interest us, since in such case it is not optimal to hold on to the perpetual option (see for example [9] and [13]). The assumption that \( C < \frac{S_T(c(\alpha))^{1+\gamma}}{\gamma S_0} \) ensures that we have \( h \leq 1 \), which means that our use of (32) was justified. The formula (35) follows from (32).

Now we turn to the comparison of results between hedging with European and American options on a numerical example. We follow the same initial data as in Sections 3.3 and 4.1. For our example we choose the hedging expenditure \( C = 0.5 \). To find the optimal strike price for VaR hedging for the expiry date, when using American put options, one has to consider numerically (32), with \( t = T \), for various strike prices. For finding the price of the American put, we use the method of approximation of the Black Scholes model by a binomial model. We then price the put using the method of super hedging by Snell envelope of the put’s payoff (For the details on the method see Lamberton and Lapeyre [13], and for the proof of the convergence in the case of American options see Kushner [12]). For this particular example, we have used a binomial tree model with four steps per day for our approximation. Down to a one cent accuracy, the optimal strike price for the American option turns out to be \( K_{opt} = 87.61 \). The price of the American option is \( P_A(K_{opt}) = 0.81 \), which is slightly higher than the price of the European put 0.74. When hedging with a perpetual American option, by Proposition 13, the optimal strike price is 99.77, which from (33) results in the price of the put equal to 7.36. Using (32) and (35), we can now draw a graph of the value at risk for the case of the American option and the perpetual put, for various termination dates \( t < T \), and compare with the results for the European options. This is done in Figure 11.

The fact that the perpetual put proves to be the least effective is not surprising, since the high price of the option allows only for a small fraction of the stock to be hedged. When comparing the American and the European cases, for times close to the maturity of the options, the European option turns out to be better (see Figure 12). This is because the European option is cheaper, which allows us to buy more options per share of stock. The overall difference between the results obtained by hedging with European and american options turns out to be negligible.
Figure 11: The VaR at termination dates $t \leq T$ for hedging with various types of options, for the hedging expenditure $C = 0.5$.

Figure 12: The difference between the $VaR_{\alpha t}$ when hedging with American and European options.
5 Conclusion

In this paper we have given an extension of the results of Ahn, Boudoukh, Richardson and Whitelaw [1, 2], for the reduction of VaR using European put options. The results of Ahn et al. hold only for a relatively small hedging expenditure and reduce the VaR of the position in stock down to a certain limit. We have shown that if one chooses a larger hedging expenditure, then the optimal strategy is to choose the strike price of the option so that its price is equal to the hedging expenditure. In other words, it is optimal to buy a single put option with an appropriate strike price. We have shown that such a strategy can be used to reduce the VaR arbitrarily close to zero. We have demonstrated that put options are more effective at reducing VaR than hedging with forward contracts. We have also investigated the impact on the VaR of the optimal position, if the investor wishes to terminate the investment prior to the expiry date of the option. It turns out that buying a European put also reduces the VaR before the terminal date since the put can be sold on the market. What is more hedging with European options produces just as good results as the ones attainable with American options.

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References


